Kuwait University **Math 102** **Date: March 25, 2006**
Dept. of Math. & CS **1st Exam** **Duration: 90 min.**

Calculators, mobile phones, pagers and all other mobile communication equipment are not allowed.

Answer the following questions:

1. Consider [2 pts. each]

$$f(x) = \tan^{-1}x + \cosh x, \qquad x \geq 0.$$

   a. Prove that f is one-to-one.

   b. Find the slope of the tangent line to the curve $y = f^{-1}(x)$ at the point $(1,0)$.

2. Prove that

$$\ln(x^r) = r\ln x$$

for any $x > 0$ and r any rational number. [2 pts.]

3. Find all solutions for the equation: [2 pts.]

$$\log_6 x + \log_6(x-5) = 1$$

4. Find the exact value of $\cos(\sin^{-1}(\frac{1}{2}) + \cos^{-1}(\frac{4}{5}))$ [2 pts.]

5. Find $\frac{dy}{dx}$ if [3 pts. each]

   a. $y = (1+x^2)^x$

   b. $ye^{xy} - \ln(1+x^2y^2) = 3y - 6x^5$

6. Evaluate each of the following integrals: [3 pts. each]

   a. $\int \frac{4x^3+3}{2x^4+6x+1}\,dx$

   b. $\int 2^{-x}e^{x+3}\,dx$

   c. $\int \frac{\sinh x}{5+\sinh^2 x}\,dx$

1. $f(x) = \tan^{-1} x + \cosh x, \quad x \geq 0$

a) $f'(x) = \dfrac{1}{1+x^2} + \sinh x > 0$

so f is increasing $\Rightarrow$ f is 1-1.

b) slope $= \dfrac{d}{dx} f^{-1}(x)\Big|_{x=1} = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = 1$

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2. Let $f(x) = \ln x^r$ and $g(x) = r \ln x$.

Then $f'(x) = \cdots = \dfrac{r}{x} = g'(x)$, so $f(x) = g(x) + C$.

$f(1) = g(1) = 0 \Rightarrow C = 0$, so $f(x) = g(x)$.

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3. $\log_6 x + \log_6 (x-5) = 1$

$\Rightarrow \log_6 x(x-5) = \log_6 6$

$\Rightarrow x(x-5) = 6 \Rightarrow (x-6)(x+1) = 0$

$\Rightarrow x = 6$ or -1, but the equation is undefined for $x = -1$; so $\underline{\underline{x = 6}}$

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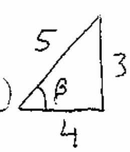
4. $\cos\left(\sin^{-1}\frac{1}{2} + \arccos\frac{4}{5}\right) =: \cos(\alpha + \beta)$

$= \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$= \dfrac{\sqrt{3}}{2}\,\dfrac{4}{5} - \dfrac{1}{2}\,\dfrac{3}{5} = \dfrac{4\sqrt{3} - 3}{10}$

$\alpha = \dfrac{\pi}{6}$



<u>5a</u> $y = (1+x^2)^x \Rightarrow \ln y = x \ln(1+x^2)$

$\Rightarrow \dfrac{y'}{y} = \ln(1+x^2) + x\,\dfrac{2x}{1+x^2}$

$\Rightarrow y' = (1+x^2)^x \left[\ln(1+x^2) + \dfrac{2x^2}{1+x^2}\right]$

5b $y e^{xy} - \ln(1 + x^2 y^2) = 3y - 6x^5$

$$\Rightarrow y' e^{xy} + y e^{xy}(y + xy') - \frac{2xy^2 + 2x^2 y y'}{1 + x^2 y^2} \quad (1)$$

$$= 3y' - 30x^4$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2 e^{xy} + \frac{2xy^2}{1+x^2y^2} - 30x^4}{e^{xy} + xy e^{xy} - \frac{2x^2 y}{1+x^2y^2} - 3} \quad (2)$$

6a $$\int \frac{4x^3+3}{2x^4+6x+1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|2x^4+6x+1| + C$$

$u = 2x^4+6x+1$ (1) (1) (1)

6b $$\int 2^{-x} e^{x+3}\,dx = e^3 \int \left(\frac{e}{2}\right)^x dx = \frac{e^3 \left(\frac{e}{2}\right)^x}{\ln\left(\frac{e}{2}\right)} + C$$

(1) (1)

$$= \frac{2^{-x} e^{x+3}}{1 - \ln 2} + C \quad (1)$$

6c $$\int \frac{\sinh x}{5 + \sinh^2 x}\,dx = \int \frac{\sinh x}{4 + \cosh^2 x}\,dx$$ (1) $u = co$ $\frac{du}{dx} = s$

$$= \int \frac{du}{4+u^2} = \frac{1}{2}\tan^{-1}\frac{u}{2} + C = \frac{1}{2}\tan^{-1}\left(\frac{\cosh x}{2}\right) + C$$

(1) (1)